A CONTINUUM OF PAIRWISE NONISOMORPHIC SPACES OF WHITNEY FUNCTIONS ON CANTOR-TYPE SETS

Alexander P. Goncharov, Mefharet Kocatepe

Department of Mathematics, Faculty of Science
Bilkent University, 06533 Bilkent, Ankara, Turkey
goncha@fen.bilkent.edu.tr
kocatepe@fen.bilkent.edu.tr

Abstract. The properties Ω_2 of Zahariuta (or $\underline{DN}$ of Vogt) and D_φ for the space $\mathcal{E}(K)$ of Whitney functions on Cantor type sets are considered. We give a criterion for $\mathcal{E}(K)$ to have these properties for two cases. As application, it is shown that there is a continuum of such spaces which are pairwise non-isomorphic.

I. Introduction.

Let $K \subset \mathbb{R}$ be a perfect compact set. By $\mathcal{E}(K)$ we denote the space of infinitely differentiable Whitney functions on K. This is the space of functions $f : K \to \mathbb{R}$ extendable to C^∞-functions on $\mathbb{R}$ equipped with the topology defined by the sequence of norms

$$\|f\|_q = |f|_q + \sup\left\{\frac{|(R_y^q f)^{(i)}(x)|}{|x-y|^{q-i}} : \, x, y \in K, \, x \neq y, \, i \leq q\right\}, \, q = 0, 1, \ldots$$

where $|f|_q = \sup\{|f^{(j)}(x)| : x \in K, j \leq q\}$ and

$$R_y^q f(x) = f(x) - T_y^q f(x) = f(x) - \sum_{k=0}^{q} \frac{f^{(k)}(y)}{k!}(x-y)^k$$

is the Taylor remainder. With

$$U_q = \{f \in \mathcal{E}(K) : \|f\|_q \leq 1\}$$

the sequence (U_q) needs not to decrease, but the sets εU_q with $\varepsilon > 0$ and $q \in \mathbb{N}$ constitute a basis of neighborhoods of zero in $\mathcal{E}(K)$. It was shown in [8] by Tidten and in [12] by Vogt that the space $\mathcal{E}(K)$ is isomorphic to the space

$$s = \left\{x = (\xi_n) : \|x\|_q = \sum_{n=1}^{\infty} |\xi_n| n^q < \infty, \, \forall q\right\}$$

of rapidly decreasing sequences if and only if there is a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R})$. By $\mathcal{E}^r(K)$ we denote the Banach space of r times differentiable Whitney jets on K equipped with the norm $\|\cdot\|_r$.

We denote $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$. We will consider Cantor type compact sets which are described as follows: let $l = (l_n)_{n=0}^\infty$ be a sequence of positive numbers and $\mathcal{N} = (N_n)_{n=1}^\infty$ be a sequence of integers $N_n \geq 2$ such that $l_0 = 1$ and $N_n l_n < l_{n-1}$ as well as $4l_n \leq l_{n-1}$. Let $K = K(l, \mathcal{N})$ be the Cantor type set $K = \cap_{n=0}^\infty K_n$ where $K_0 = [0, 1] = [0, l_0]$ and K_1 is obtained from K_0 by deleting $N_1 - 1$ equal open intervals of total length $l_0 - N_1 l_1$ uniformly. Thus K_1 is the union of N_1 closed intervals $I_{1,k}$ each of length l_1 which are distributed uniformly. In turn K_n is a union of $N_1 N_2 \dots N_n$ closed intervals $I_{n,k}$ of length l_n and K_{n+1} is obtained from K_n by deleting uniformly $N_{n+1} - 1$ equal open intervals of total length $l_n - l_{n+1} N_{n+1}$ from each $I_{n,k}$, $k = 1, 2, \dots, N_1 N_2 \dots N_n$. This way for the classical Cantor set we have $l_n = 3^{-n}$ and $N_n = 2$. We will consider two cases: (i) $N_n = 2$ for all n, (ii) $\lim_{n\to\infty} N_n = \infty$.

The following lemma is in [4] (see also [5]).

Lemma 1. Let $g(x) = \prod_{j=1}^N (x - a_j)$, where $|x - a_j| \leq l < 1, j = 1, \dots, N$. Let $f(x) = g(x)^q$. Then for $n \leq N \cdot q$

$$|f^{(n)}(x)| \leq C(N, q, n)\, l^{N\cdot q - n},$$

If in addition $n < q$, then

$$|f^{(n)}(x)| \leq C(N, q, n) \cdot |g(x)|^{q-n},$$

where

$$C(N, q, n) = \frac{(N \cdot q)!}{(N \cdot q - n)!}.$$

The following lemma is a variation of Lemma 2 in [5] with a very similar proof. Hence the proof is omitted.

Lemma 2. Let $K \subset \mathbb{R}$ be a compact set containing $r + 1$ distinct points $x_0, x_1, \dots, x_r$ such that for some finite sequence $0 < \psi_1 \leq \psi_2 \leq \cdots \leq \psi_r$,

$$\frac{\psi_i}{M} \leq |x_i - x_k| \leq \psi_i \text{ for } k = 0, 1, \dots, i - 1, \;\; i = 1, 2, \dots, r.$$

Then for all $k \leq r$ and $f \in \mathcal{E}^r(K)$

$$|f^{(k)}(x_0)| \leq C\psi_1^{-k}|f|_0 + C\psi_r^{r-k}\|f\|_r$$

where the constant $C = 2kM^r r!/(r - k)!$.

Now we consider a linear topological invariant introduced by Vogt [13] and Tidten [9] (and called DN_φ by them) and by Goncharov and Zahariuta [2], [19] and [6].

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function, $\varphi(t) \geq t$. A Fréchet space X with a fundamental increasing system of seminorms $(\|\cdot\|_p)_{p=0}^{\infty}$ has the property D_φ if $\exists p \in \mathbb{Z}_+ \ \forall q \in \mathbb{N} \ \exists r \in \mathbb{N}, m > 0, C > 0$ such that

$$\|f\|_q \leq \varphi^m(t)\|f\|_p + \frac{C}{t}\|f\|_r, \ \ t > 0, \ f \in X. \tag{1}$$

We will use as well the multiplicative version of this property (see e.g. [6]):

$$\exists p \ \forall q \ \exists r, m, C : \ \frac{\|f\|_p}{\|f\|_q}\varphi^m\left(C\frac{\|f\|_r}{\|f\|_q}\right) \geq 1, \ \ \forall f \neq 0.$$

Examples of continua of pairwise non-isomorphic spaces of C^∞ ([9], [6]) and Whitney ([7], [5]) functions were found by means of these invariants.

The invariant D_φ appeared as a generalization of the class D_1 (see [15]) or the property DN ([10]). In the case $\varphi(t) = t$ the invariant D_φ coincides with Ω_2 ([18]) or $\underline{DN}$ ([11]). The strong form of the condition (1) with $m = 1$ coincides with the property DN_φ ([13], [9]).

We define $\alpha_n = \frac{\ln l_{n+1}}{\ln l_n}$. Thus $\alpha_n > 1$ and $l_{n+1} = l_n^{\alpha_n}$. In [4] the case $\alpha_n = \alpha \neq 2$ was considered. Applying Vogt-Tidten's [8] characterization of the extension property in terms of the invariant DN, it was proved in [4] that for the space $\mathcal{E}(K)$ there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R})$ if and only if $\alpha < 2$.

II. The first model case: bounded (N_n).

First let us consider the case in which the sequence (N_n) is bounded. Without loss of generality we assume that $N_n = 2$ since for arbitrary bounded sequence (N_n) we can repeat the proof with minor modifications. For the weak property $\underline{DN}$ we have the following geometric characterization.

Theorem 1. Let $N_n = 2$ for all n. The space $\mathcal{E}(K)$ has property Ω_2 (or $\underline{DN}$) if and only if $\limsup \alpha_n < \infty$.

Proof. *Necessity.* The proof of this part is similar to the proof of Theorem 2 in [4] We assume that $\limsup \alpha_n = \infty$ and show that

$$\forall p \ \exists q \ \forall r, R \ \exists (f_k) \subset \mathcal{E}(K) : \ \lim_{k\to\infty} \frac{\|f_k\|_p}{\|f_k\|_q}\left(\frac{\|f_k\|_r}{\|f_k\|_q}\right)^R = 0.$$

Fix p, let $q = p + 1$. Fix r, R such that $q < r$. Find s such that $2^{s-1} < \frac{r}{q} \leq 2^s$. Since $\limsup \alpha_n = \infty$, there is a subsequence (n_k) of integers such that $\alpha_{n_k} \nearrow \infty$. Fix k large enough and let $n = n_k + 1$. Consider the first 2^s intervals of K_n; $I_{n,1} = [0, l_n], I_{n,2} = [l_{n-1} - l_n, l_{n-1}], \ldots, I_{n,2^s} = [l_{n-s} - l_n, l_{n-s}]$. Let c_j be the midpoint of $I_{n,j}, j = 1, 2, \ldots, 2^s$. Set $f_k(x) = (g(x))^q$ where

$$g(x) = \begin{cases} \prod_{j=1}^{2^s}(x - c_j) & \text{if } x \in K \cap [0, l_{n-s}] \\ 0 & \text{otherwise.} \end{cases}$$

Upper bound of $\|f_k\|_p$. Fix $i \le p$. As in [4], $|f_k^i(x)| \le C_p(l_n\,\lambda)^{q-i}$ and $|f_k|_p \le C_p(l_n\lambda)^{q-p} = C_p l_n \lambda$ where $\lambda = l_{n-1} l_{n-2}^2 \cdots l_{n-s}^{2^{s-1}}$. With

$$A_p := \frac{|(R_x^p f_k)^{(i)}(y)|}{|x-y|^{p-i}},\ i \le p,\ x \ne y,\ x, y \in K$$

we have $A_p \le 2C_p l_n \lambda$ if $|x-y| < l_{n-1} - 2l_n$. If $|x-y| \ge l_{n-1} - 2l_n$, then

$$\begin{aligned} A_p &\le \frac{|f_k^{(i)}(y)|}{|x-y|^{p-i}} + \sum_{t=i}^{p} \frac{|f_k^{(t)}(x)|}{(t-i)!}\,\frac{1}{|x-y|^{p-t}} \\ &\le C_p(l_n\,\lambda)^{q-p}\left\{\left(\frac{l_n\,\lambda}{|x-y|}\right)^{p-i} + \sum_{t=i}^{p} \frac{1}{(t-i)!}\left(\frac{l_n\,\lambda}{|x-y|}\right)^{p-t}\right\}. \end{aligned}$$

Since $\alpha_{n-1} = \alpha_{n_k}$ is large and therefore $3l_n \le l_{n-1}$ we have that $l_n(\lambda+2) < 3l_n \le l_{n-1}$, so $l_n\,\lambda < l_{n-1} - 2l_n \le |x-y|$ which implies $l_n\,\lambda/|x-y| < 1$. Thus the expression in $\{\ \}$ above is less than $1+e$. Thus $A_p \le C_p(1+e)(l_n\,\lambda)^{q-p}$ and $\|f_k\|_p \le 5C_p(l_n\,\lambda)^{q-p} = 5C_p l_n \lambda$. Since $3l_n \le l_{n-1}$, we have $l_n < l_{n-1}/2$, and so by [4] Theorem 2, $\|f_k\|_q \ge q!\,2^{-2r}\,\lambda^q$ and $\|f_k\|_r \le C_r$.
Thus for some constant M we have

$$\frac{\|f_k\|_p}{\|f_k\|_q}\left(\frac{\|f_k\|_r}{\|f_k\|_q}\right)^R \le M\frac{l_n\lambda}{\lambda^q}\frac{1}{\lambda^{qR}} = M\frac{l_n}{\lambda^C} =: Q$$

where $C = p + (p+1)R$. We have $l_n = l_{n-1}^{\alpha_{n-1}} = \ldots = l_{n-s}^{\alpha_{n-s}\cdots\alpha_{n-2}\alpha_{n-1}}$ and so $\lambda = l_{n-s}^{\kappa}$ where

$$\kappa = \alpha_{n-2}\alpha_{n-3}\cdots\alpha_{n-s} + 2\alpha_{n-3}\cdots\alpha_{n-s} + \cdots + 2^{s-2}\alpha_{n-s} + 2^{s-1}.$$

So we get $Q = M\,l_{n-s}^{\alpha_{n-1}\alpha_{n-2}\cdots\alpha_{n-s} - C\kappa}$. Observe that κ is a sum of s terms and each of them is less than $2^{s-1}\alpha_{n-2}\cdots\alpha_{n-s}$. Then

$$\alpha_{n-1}\alpha_{n-2}\ldots\alpha_{n-s} - C\kappa = \alpha_{n-2}\ldots\alpha_{n-s}(\alpha_{n-1} - Cs2^{s-1}) \to \infty$$

as s is fixed. Thus the exponent of l_{n-s} tends to ∞ and so $Q \to 0$ as $n \to \infty$.
Sufficiency. If $\limsup \alpha_n < \infty$, then for some $\alpha < \infty$ we have $\alpha_n \le \alpha$ for all n.
Let $p = 0$. Given q, let $q_1 = 2q,\ r = 4q$ and $R = \alpha^r + 1$. Fix $t > 1$ and find n such that $l_{n-r+1} \le 1/t < l_{n-r}$. Fix $x_0 \in K$ and find intervals $I_{m,j} \subset K_m$ such that

$$x_0 \in I_{n,j_0} \subset I_{n-1,j_1} \subset \ldots \subset I_{n-(r-1),j-(r-1)}.$$

Let x_1 be the end point of I_{n,j_0} which is farther from x_0. Then clearly $l_n/2 \le |x_1 - x_0| \le l_n$. Let x_2 be the end point of I_{n-1,j_1} which is farther from x_0. Then

$$\frac{l_{n-1}}{2} \le |x_2 - x_0| \le l_{n-1} \text{ and } \frac{l_{n-1}}{2} \le |x_2 - x_1| \le l_{n-1}$$

since x_1 and x_2 belong to different intervals $I_{n,j}$ and $I_{n,j'}$ in I_{n-1,j_1}. We continue in this way and choose the points $x_1, x_2, \ldots, x_r$. Then

$$\frac{l_{n-i+1}}{2} \leq |x_i - x_k| \leq l_{n-i+1}, \; k = 0, 1, \ldots, i-1; \; i = 1, 2, \ldots, r.$$

So we can apply Lemma 2 with $\psi_i = l_{n-i+1}$ and for all $k \leq q_1$ obtain

$$|f^{(k)}(x_0)| \leq C l_n^{-k} |f|_0 + C l_{n-r+1}^{r-k} \|f\|_r.$$

Since $l_n = l_{n-1}^{\alpha_{n-1}} = \ldots = l_{n-r}^{\alpha_{n-1} \ldots \alpha_{n-r}} \geq l_{n-r}^{\alpha^r} \geq t^{-\alpha^r}$, for all $k \leq q_1$ we have

$$\begin{aligned} |f^{(k)}(x_0)| t^{q_1 - k} &\leq C_1 t^{\alpha^r k + q_1 - k} |f|_0 + \frac{C_2}{t^{r-q_1}} \|f\|_r \\ &\leq C_1 t^{Rq} |f|_0 + \frac{C_2}{t^q} \|f\|_r =: S(t). \end{aligned}$$

Then as in the last paragraph of the proof of Theorem 3 in [4] we have $\|f\|_q \leq C_0 S(t)$, which proves the theorem.

□

In the next theorem we give a necessary and sufficient condition for the space $\mathcal{E}(K)$ to have the property D_φ

Theorem 2. Assume $N_n = 2$ for all n and $\lim_{n\to\infty} \alpha_n = \infty$. Then $\mathcal{E}(K)$ has property D_φ if and only if the following condition is true:

$$\forall k \; \exists M : \quad l_n \geq \varphi^{-M}(l_{n-k}^{-1}) \;\; \forall n.$$

Proof. *Necessity.* By D_φ we have p. Given k, let $q = 2^{k+1}(p+1) - 1$ and by D_φ we choose $r \in \mathbb{N}$, $M \geq 1$ and $C \geq 1$ such that for all $f \in \mathcal{E}(K)$, $f \neq 0$ we have

$$\frac{\|f\|_p}{\|f\|_q} \varphi^M \left(C \frac{\|f\|_r}{\|f\|_q} \right) \geq 1.$$

Let s be such that $2^{s-1} < r \leq 2^s$. Fix n large enough and consider the first 2^s intervals of K_n; $I_{n,1} = [0, l_n], I_{n,2} = [l_{n-1} - l_n, l_{n-1}], \ldots, I_{n,2^s} = [l_{n-s} - l_n, l_{n-s}]$. Let c_j be the midpoint of $I_{n,j}, j = 1, 2, \ldots, 2^s$. Set $f_n(x) = (g(x))^{p+1}$ where

$$g(x) = \begin{cases} \prod_{j=1}^{2^s} (x - c_j) & \text{if } x \in K \cap [0, l_{n-s}] \\ 0 & \text{otherwise.} \end{cases}$$

Then as in Theorem 1, we have $\|f\|_p \leq 5C_p (l_n \lambda)^{p+1-p} \leq l_n$ for large enough n and $\|f\|_r \leq C_r$.

Lower bound of $\|f_n\|_q$. We have $\|f_n\|_q \geq |f_n|_q \geq |f_n^{(q)}(c_{2^s})|$ and

$$f_n(x) = \prod_{j=1}^{2^s} (x - c_j)^{p+1} = \prod_{i=1}^{(p+1)2^s} (x - b_i)$$

where

$$b_{j(p+1)+1} = b_{j(p+1)+2} = \ldots = b_{(j+1)(p+1)} = c_{j+1}, \quad j = 0, 1, \ldots, 2^s - 1.$$

Then

$$f^{(q)}(x) = \sum_{\mathrm{Card}(A)=(p+1)2^s-q} C(A) \prod_{i\in A}(x - b_i)$$

where $A \subset \{1, 2, \ldots, 2^s(p+1)\},\ C(A) \geq 1$. Then each term in $f^{(q)}(b_{(p+1)2^s}) = f^{(q)}(c_{2^s})$ is nonnegative. So

$$|f^{(q)}(c_{2^s})| \geq (c_{2^s} - b_1)(c_{2^s} - b_2)\cdots(c_{2^s} - b_{(p+1)2^s-q}) =: B.$$

Since $(p+1)2^s - q = (p+1)2^s - (2^{k+1}(p+1) - 1) = (p+1)(2^s - 2^{k+1}) + 1$, we have

$$B = (c_{2^s} - c_1)^{p+1}(c_{2^s} - c_2)^{p+1}\cdots(c_{2^s} - c_{2^s-2^{k+1}})^{p+1}(c_{2^s} - c_{2^s-2^{k+1}+1}).$$

Note that $2^s - 2^{k+1} = 2^{s-1} + 2^{s-2} + \cdots + 2^{k+1}$ and

$$\begin{aligned}
c_{2^s} - c_1 > c_{2^s} - c_2 > \cdots > c_{2^s} - c_{2^{s-1}} = l_{n-s} - l_{n-(s-1)} &> \frac{l_{n-s}}{2}\\
c_{2^s} - c_{2^{s-1}+1} > \cdots > c_{2^s} - c_{2^{s-1}+2^{s-2}} = l_{n-(s-1)} - l_{n-(s-2)} &> \frac{l_{n-(s-1)}}{2}\\
\cdots\\
c_{2^s} - c_{2^{s-1}+\ldots+2^{k+2}+1} > \cdots > c_{2^s} - c_{2^{s-1}+\ldots+2^{k+2}+2^{k+1}} &> \frac{l_{n-(k+2)}}{2}\\
c_{2^s} - c_{2^s-2^{k+1}+1} = c_{2^s} - c_{2^{s-1}+\ldots+2^{k+2}+2^{k+1}+1} &> \frac{l_{n-(k+1)}}{2}.
\end{aligned}$$

Thus

$$\|f\|_q \geq B \geq C l_{n-k-1}\left(l_{n-k-2}^{2^{k+1}}\cdots l_{n-s}^{2^{s-1}}\right)^{p+1} \geq C l_{n-k-1}^{2^s(p+1)} \geq \frac{1}{K} l_{n-k}$$

for large n and any constant K since $l_{n-k} = l_{n-k-1}^{\alpha_{n-k-1}}$ and $\lim_{n\to\infty} \alpha_{n-k-1} \to \infty$. Next we choose K large enough so that the second inequality below holds:

$$1 \leq \varphi^M\left(C\frac{\|f\|_r}{\|f\|_q}\right)\frac{\|f\|_p}{\|f\|_q} \leq \varphi^M\left(\frac{1}{l_{n-k}}\right)\cdot\frac{l_n\lambda}{l_{n-k}} \leq \varphi^M\left(\frac{1}{l_{n-k}}\right)\cdot\frac{l_n}{l_{n-k}} \leq \varphi^{M+1}\left(\frac{1}{l_{n-k}}\right) l_n.$$

Thus $l_n \geq \varphi^{-(M+1)}(l_{n-k}^{-1})$ for all large n. By enlarging $M+1$ if necessary, we have this inequality for all n.

Sufficiency. Suppose that

$$\forall r\ \exists M_r:\ l_n \geq \varphi^{-M_r}(l_{n-r}^{-1}),\ \forall n.$$

Let us take $p = 0$. Given q, let $q_1 = 2q,\ r = 2q+1$ and $M = (M_r+1)2q$, where M_r is defined by the condition above. Fix t and n such that $l_{n-r+1} \leq 1/t < l_{n-r}$. Fix $f \in \mathcal{E}(K)$.

We can now proceed in a way analogous to the proof of Theorem 1. For $x_0 \in K,\ k \leq q_1$ we have

$$\begin{aligned} |f^{(k)}(x_0)| &\leq C_1 l_n^{-k} |f|_0 + C_2 l_{n-r+1}^{r-k} \|f\|_r \\ &\leq C_1 \varphi^{M_r k}(t) |f|_0 + C_2 t^{k-r} \|f\|_r. \end{aligned}$$

Therefore,

$$\begin{aligned} |f^{(k)}(x_0)|\, t^{q_1-k} &\leq C_1 \varphi^{(M_r+1)q_1}(t) |f|_0 + C_2 t^{q_1-r} \|f\|_r \\ &= C_1 \varphi^{M}(t) |f|_0 + \frac{C_2}{t} \|f\|_r. \end{aligned}$$

¿From here it is easy to obtain the desired bound

$$\|f\|_q \leq C_1' \varphi^{M}(t) |f|_0 + \frac{C_2'}{t} \|f\|_r$$

where the constants $C_1',\ c_2'$ depend on $t,\ f$. Thus the space $\mathcal{E}(K)$ has the property D_φ. □

III. The second model case: unbounded (N_n)**.**
Next we consider a compact set $K = K(l, \mathcal{N})$ where $\lim_{n\to\infty} N_n = \infty$. We write

$$K_n = I_{n,1} \cup I_{n,2} \cup \cdots I_{n,N_n} \cup I_{n,N_n+1} \cup \cdots \cup I_{n,N_n N_{n-1} \ldots N_1}$$

where the intervals above are pairwise disjoint. Let us denote the distance between $I_{n,1}$ and $I_{n,2}$ by h_n.
Theorem 3. Assume $K = K(l, \mathcal{N})$ where $\lim_{n\to\infty} N_n = \infty$, $l_n < h_n$ and for some $Q \geq 1$, $h_n \geq l_{n-1}^Q$ for all n. Then $\mathcal{E}(K)$ has D_φ if and only if the following condition is true:

$$\exists M :\ \ l_n \geq \varphi^{-M}(l_{n-1}^{-M}),\ \forall n. \tag{2}$$

Proof. We will consider the condition

$$\exists M :\ \ l_n \geq \varphi^{-M}(h_n^{-M}),\ \forall n$$

which is clearly equivalent to (2).
Necessity. By D_φ we have p. Let $q = p + 1$ and find r, R, C such that for all $f \in \mathcal{E}(K)$ we have

$$1 \leq \frac{\|f\|_p}{\|f\|_q} \left(C \frac{\|f\|_r}{\|f\|_q} \right)^R.$$

Now given n, define

$$f_n(x) = \begin{cases} \frac{x^q}{q!} & \text{if } x \in K \cap [0, l_n] \\ 0 & \text{otherwise} \end{cases}$$

Then it can be easily shown as in the previous theorems that $\|f\|_p \le 4l_n,\ \|f\|_q \ge 1,\ \|f\|_r \le 4h_n^{q-r}$. Thus the inequality above holds for $M > \max\{R, r-q\}$.
Sufficiency. Let $p = 0$. Given q let $r = q+2$ and $m = \max\{QM(q+1), (QM+1)q\}$. We will show that there are constants $\tilde{C}_1$ and $\tilde{C}_2$ such that for all $f \in \mathcal{E}(K)$

$$\|f\|_q \le \tilde{C}_1 \varphi^m(t^{M+1})|f|_0 + \frac{\tilde{C}_2}{t}\|f\|_r,\ \forall t > 0.$$

This is D_φ since $M+1$ does not depend on q (see e.g. [3].)
Let n_0 be such that for all $n \ge n_0$ we have $2r < N_n$. Given $t \ge t_0 := \max\{2^M r^M, 1/l_{n_0-1}\}$, we find n such that $l_n < 1/t \le l_{n-1}$. We will apply Lemma 2 in [5]. Let $x_0 \in K$. Then $x_0 \in I_{n,j_0}$. To simplify writing we may assume that $1 \le j_0 \le N_n$.
Case 1. $1/t \ge a$ where a is the left end point of $I_{n,r+1}$.
(i) If $j_0 \le N_n/2$, we choose x_μ as the left end point of $I_{n,j_0+\mu}$. Then $x_0 < x_1 < \cdots < x_r$ and $h = x_1 - x_0 \le x_2 - x_1 = \cdots = x_r - x_{r-1} = H$, and so by Lemma 2 in [5] we have for $k \le r$

$$|f^{(k)}(x_0)| \le C_1 h^{-k}|f|_0 + C_2 H^{r-k}\|f\|_r.$$

Since $h \ge h_n \ge l_{n-1}^Q \ge 1/t^Q$ and $H = h_n + l_n \le a \le 1/t$ we have

$$|f^{(k)}(x_0)| \le C_1 t^{Qk}|f|_0 + \frac{C_2}{t^{r-k}}\|f\|_r \le C_1 \varphi^{QMk}(t^{M+1})|f|_0 + \frac{C_2}{t^{r-k}}\|f\|_r.$$

(ii) If $j_0 > N_n/2$, then we choose x_μ as the right end point of $I_{n,j_0-\mu}$. Then $x_0 > x_1 > \cdots > x_r$, but Lemma 2 in [5] can be applied and we may proceed as in (i).
Case 2. $1/t < a$. Then $l_n < 1/t < a$. In this case we choose all the points $x_1, x_2, \dots, x_r$ in I_{n,j_0}. Since I_{n,j_0} is the union of N_{n+1} intervals $I_{n+1,i}$ and $x_0 \in I_{n+1,i_0}$ for some i_0, we can choose $x_\mu \in I_{n+1,i_0+\mu}$ for all $\mu = 1, 2, \dots, r$ or $x_\mu \in I_{n+1,i_0-\mu}$ for all $\mu = 1, 2, \dots, r$. Then arguing as above, we see that

$$|f^{(k)}(x_0)| \le C_1|x_1 - x_0|^{-k}|f|_0 + C_2|x_r - x_{r-1}|^{r-k}\|f\|_r.$$

Since $|x_1 - x_0| \ge h_{n+1} \ge l_n^Q \ge \varphi^{-QM}(h_n^{-M})$ and from $a = r(h_n + l_n) \le 2rh_n$ we get $h_n \ge a/(2r) > 1/(2rt)$ we get $|x_1 - x_0|^{-k} \le \varphi^{QMk}(2^M r^M t^M) \le \varphi^{QMk}(t^{M+1})$. Also $|x_r - x_{r-1}| \le l_n \le 1/t$. Thus for all $k \le r$ and for all $t \ge t_0$ we have

$$|f^{(k)}(x_0)| \le C_1 \varphi^{QMk}(t^{M+1})|f|_0 + \frac{C_2}{t^{r-k}}\|f\|_r.$$

Next we estimate

$$A_q = \frac{|(R_x^q f)^{(k)}(y)|}{|x-y|^{q-k}},\ x, y \in K,\ x \ne y,\ k \le q.$$

Given $x, y \in K,\ x \neq y$ and $t \geq t_0$, if $|x-y| \geq 1/t$, then

$$\begin{aligned} A_q &\leq \frac{|f^{(k)}(y)|}{|x-y|^{q-k}} + \sum_{i=k}^{q} \frac{|f^{(i)}(x)|}{(i-k)!} \frac{1}{|x-y|^{q-i}} \\ &\leq C_1 \varphi^{QMk}(t^{M+1})|f|_0 t^{q-k} + \frac{C_2}{t^{r-k}} \|f\|_r t^{q-k} \\ &+ \sum_{i=k}^{q} C_1 \varphi^{QMi}(t^{M+1})|f|_0 \frac{t^{q-i}}{(i-k)!} + \sum_{i=k}^{q} \frac{C_2}{t^{r-i}} \|f\|_r \frac{t^{q-i}}{(i-k)!} \\ &\leq C_1 \varphi^{QMq+q}(t^{M+1})|f|_0(1+e) + \frac{C_2}{t^{r-q}} \|f\|_r (1+e). \end{aligned}$$

If $|x-y| < 1/t$, then from

$$R_x^q f(y) = R_x^{q+1} f(y) + f^{(q+1)}(x) \frac{(y-x)^{q+1}}{(q+1)!}$$

it follows that

$$\begin{aligned} A_q &\leq \|f\|_{q+1}|x-y| + \frac{|f^{(q+1)}(x)|}{(q+1-k)!}|x-y| \\ &\leq \|f\|_r \frac{1}{t} + C_1 \varphi^{QM(q+1)}(t^{M+1})|f|_0 \frac{1}{t} + \frac{C_2}{t^{r-(q+1)}} \|f\|_r \frac{1}{t} \end{aligned}$$

Thus we have constants $\tilde{C}_1$ and $\tilde{C}_2$ such that for all $f \in \mathcal{E}(K)$

$$\|f\|_q \leq \tilde{C}_1 \varphi^m(t^{M+1})\|f\|_0 + \frac{\tilde{C}_2}{t}\|f\|_r$$

and the space $\mathcal{E}(K)$ has the property D_φ.

□

Now we can construct families having the cardinality of the continuum of pairwise nonisomorphic spaces $\mathcal{E}(K)$ for any model type.

Example 1. Let $l_1 = e^{-1},\ N_n = 2,\ \alpha_n = \exp n^\lambda$ with $\lambda > 1$ and K_λ denote the corresponding Cantor-type set. Then by Theorem 2 the space $\mathcal{E}(K_\lambda)$ has the property D_φ if and only if

$$\forall k\ \exists M:\ \varphi^M(e^{\alpha_1 \dots \alpha_n}) \geq e^{\alpha_1 \dots \alpha_{n+k}},\ \forall n. \tag{3}$$

Let us show that if $\lambda \neq \mu$ then the spaces $\mathcal{E}(K_\lambda)$ and $\mathcal{E}(K_\mu)$ are not isomorphic. Given $\lambda < \mu$ let us take ρ with $\lambda/(\lambda+1) < \rho < \mu/(\mu+1)$ and $\varphi(t) = t^{\gamma(t)}$ with $\gamma(t) = \exp \ln^\rho \ln t$. Let us show that the space $\mathcal{E}(K_\lambda)$ has the property D_φ whereas $\mathcal{E}(K_\mu)$ does not have it. Substituting the function φ in (3) gives the condition

$$\forall k\ \exists M:\ M\gamma(e^{\alpha_1 \dots \alpha_n}) \geq \alpha_{n+1} \dots \alpha_{n+k},\ \forall n$$

or

$$\ln M + (\ln \alpha_1 + \cdots + \ln \alpha_n)^\rho \geq \ln \alpha_{n+1} + \cdots + \ln \alpha_{n+k}, \ \forall n.$$

Since

$$\frac{n^{\lambda+1}}{\lambda+1} < 1 + 2^\lambda + \cdots + n^\lambda < \frac{(n+1)^{\lambda+1}}{\lambda+1}$$

and

$$kn^\lambda < (n+1)^\lambda + \cdots + (n+k)^\lambda < k2^\lambda n^\lambda \text{ if } n > k,$$

we see that for the space $\mathcal{E}(K_\lambda)$ the condition above is valid. Suppose that it is valid also for $\mathcal{E}(K_\mu)$. Then for $k = 1$ we have M_1 such that

$$\ln M_1 + \left(\frac{n+1}{\mu+1}\right)^{(\mu+1)\rho} \geq n^\mu, \ n \to \infty,$$

which is a contradiction as $\rho(\mu+1) < \mu$. Therefore $\mathcal{E}(K_\lambda) \not\simeq \mathcal{E}(K_\mu)$.

Example 2. Let $l_1, \ \alpha_n$ be the same as before but now let $h_n = l_{n-1}^2$. Then $N_n > l_{n-1}/(l_n + h_n) \to \infty$ as $n \to \infty$ and we have the compact set $K_\lambda = K((l_n), (N_n))$ satisfying the conditions of Theorem 3. For $\varphi(t) = t^{\gamma(t)}$ we get the following characterization: $\mathcal{E}(K)$ has the property D_φ if and only if

$$\exists M : \ M^2 \gamma(e^{M\alpha_1 \ldots \alpha_n}) \geq \alpha_{n+1}, \ \forall n.$$

Let us fix λ, μ, ρ and $\gamma(t)$ as before. We see that the space$\mathcal{E}(K_\lambda)$ has the property D_φ whereas $\mathcal{E}(K_\mu)$ does not have.

We guess that the invariant D_φ is complete for the spaces $\mathcal{E}(K)$ of the first type. On the other hand, for the spaces $\mathcal{E}(K)$ of the second type ($N_n \to \infty$) it is possible as in [1] to find nonisomorphic spaces which are not distinguishable by the invariant D_φ, but can be distinguished by invariants based on the methods of Zahariuta [14], [16], [17], [5].

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